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On steady flow in a partially filled rotating cylinder

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The flow in a partially filled cylinder rotating at right angles to the earth's gravity is found under the assumptions of rapid rotation and small viscosity. Effects of viscosity, nonlinear interaction and finite container length are included.

1. Introduction

Flow in a partially filled, horizontally rotating cylinder has been examined both experimentally and theoretically by a number of investigators, including Phillips (1960), Karweit & Corrsin (1975), Greenspan (1976) and Ruschak & Scriven (1976). Motives have been various, including modelling of the generation of suction spots by tornadoes (Greenspan) and engineering applications such as cream separators (Ruschak & Scriven).

The physical picture is deceptively simple: a cylinder partly filled with a fluid is spun about its symmetry axis sufficiently rapidly that the fluid is held out against the walls by the centrifugal force. If the rotation axis is horizontal there is a non-axisymmetric force on the fluid. One expects that a simple steady-state (in the laboratory frame) motion would result over some range of parameter space, and that one could even construct some simple stability criterion, relating gravity to the centrifugal force perhaps.

Those experiments and observations which have been published (Phillips 1960; Karweit & Corrsin 1975; Greenspan 1976) indicate that steady-state phenomena are often submerged in a welter of time-dependent phenomena. Many of these latter phenomena are fascinating: a time-dependent instability noted by Phillips, some streaming cell structures noted by Karweit & Corrsin and the generation of vortices noted by Greenspan.

The theoretical work to date has been inadequate for understanding the basis of these time-dependent processes. (It seems likely that a full understanding can be gained only experimentally, however some further theoretical work is necessary to enable one to interpret further experimental work.) Phillips and Ruschak & Scriven address linear problems. Greenspan's analysis is nonlinear in a perturbation sense, and is carried to second order in a small amplitude, but it neglects the finite length of the container and replaces the free surface by a mobile rigid surface.

In this paper I account for viscous effects, nonlinear effects and the effects of finite container length in an effort to find a steady-state solution which can underlie the many observed time-dependent phenomena. To that end I suppose that the Froude number $\Omega^2 a/g$ ($= 1/\epsilon$) is large and the Ekman number $\nu/\Omega a^2$ ($= E$) is small, consider an amplitude/boundary-layer combined expansion, and carry that out to $O(\epsilon^2)$ in the axisymmetric part of the solution.

The range of parameter space for which the solutions are valid is

$$0 < E^{\frac{1}{2}} \ll \epsilon \ll E^{\frac{1}{4}} \ll c < 1,$$

where c is the dimensionless radius of the air column. The major conclusions are as follows.

(i) That the air column is offset from the centre an amount $\frac{1}{2}\epsilon(1-c^2)\,a$ (which agrees with Phillips 1960) in a direction making an angle $-2(2E)^{\frac{1}{2}}c^2/(1-c^2)$ with the vertical (a new result), the sense of the displacement being downwards.

(ii) That the axisymmetric flow in the interior is dominated by a swirl proportional to the inverse fifth power of the radius.

(iii) That steady-state circulations in the interior have mass flux small compared with $E\epsilon^2$.

(iv) That Stewartson ($E^{\frac{1}{4}}$ and $E^{\frac{1}{3}}$) layers form on the inner and outer radial boundaries, and that there is circulation within the layers.

The plan of the paper is as follows. Section 2 presents a mathematical formulation, of necessity rather long. Section 3 gives the lowest-order solution, the inviscid part of which is that found by Phillips. The viscous corrections are new. Section 4 gives the rectified (axisymmetric) problem including all the relevant boundary layers and mass fluxes. Section 5 gives a brief discussion.

2. Mathematical formulation

Let a, b, L, Ω_0, ν, ρ and $\mathbf{g}$ denote the outer radius, inner radius, container length, basic rotation rate, kinematic viscosity, fluid density and the acceleration of gravity respectively. (If V denotes the volume of air in the container, b can be defined as $b = (V/\pi L)^{\frac{1}{2}}$ in circumstances under which there is no well-defined interface.)

The governing equations for $\mathbf{v}$ and P, made dimensionless by $\Omega_0 a$ and $\rho\Omega_0^2 a^2$ respectively, are

$$\mathbf{v}_t + \mathbf{v}\,.\,\nabla\mathbf{v} + \nabla P = E\nabla^2\mathbf{v} + \epsilon\boldsymbol{\gamma}, \tag{2.1}$$

$$\nabla\,.\,\mathbf{v} = 0, \tag{2.2}$$

where $E = \nu/\Omega_0 a^2$ is an Ekman number, $\epsilon = g/\Omega_0^2 a$ is an inverse Froude number and $\boldsymbol{\gamma}$ is a unit vector parallel to $\mathbf{g}$. The boundary conditions are those of no slip on the solid surfaces and vanishing stress on the free surface.† There is, in addition, a kinematic condition relating the radial velocity to the rate of change of the interface location.

Before proceeding with the rituals of linearization and expansion it is convenient to introduce two specific co-ordinate systems. These are shown in figure 1. The $\bar{x}$, $\bar{y}$ system is centred in the cylinder and will be referred to as the container system. The x, y system is centred in the air core and has its origin at a distance δ from the centre of the cylinder. It will be called the core system.

First-order inviscid theory (Phillips 1960) indicates that the air column is depressed in a direction parallel to $\boldsymbol{\gamma}$. In general one should introduce an angle α between the vector joining the two origins and $\boldsymbol{\gamma}$, making $\boldsymbol{\gamma} = -\cos\alpha\mathbf{j} + \sin\alpha\mathbf{i}$. The dimensionless radius of the air core is $c = b/a$.

Because the relationship between the two systems is a simple translation, the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the same in both, and the Cartesian components

† Perhaps a stringent condition for real fluids, but a reasonable limit.

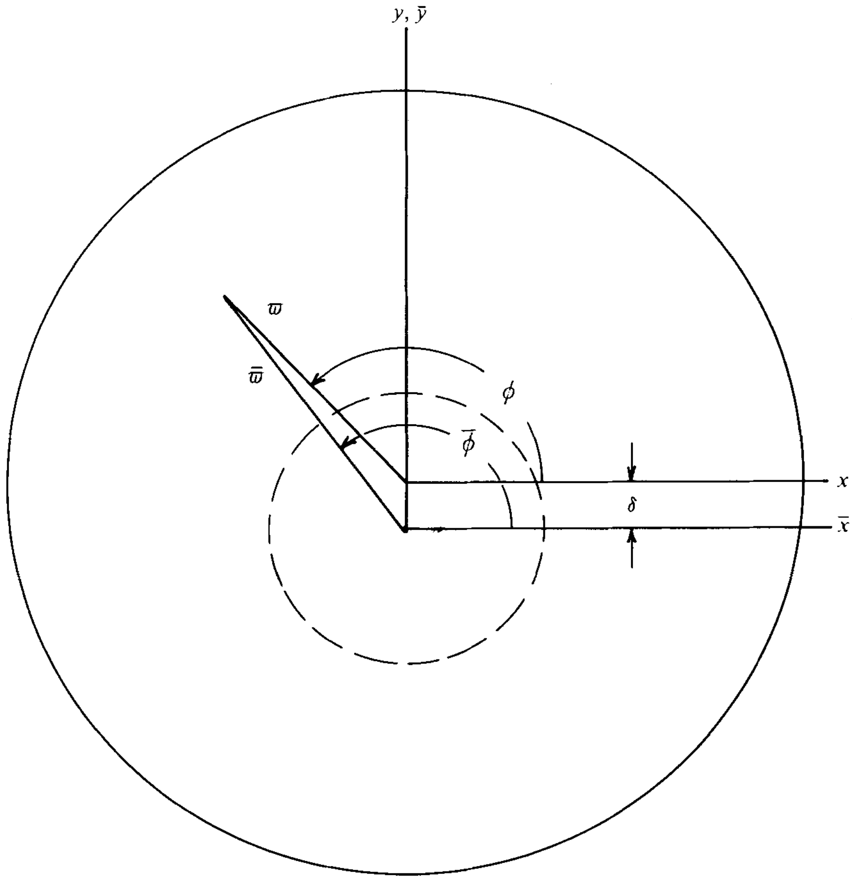

FIGURE 1. Sketch of the core and container co-ordinates.

of velocity must be the same. It is, however, more convenient to work in cylindrical co-ordinates defined by

$$x = \varpi\cos\phi,\quad y = \varpi\sin\phi,\quad \bar{x} = \bar{\varpi}\cos\bar{\phi},\quad \bar{y} = \bar{\varpi}\sin\bar{\phi}. \tag{2.3}$$

The velocity and pressure will be written in core co-ordinates as

$$\left.\begin{aligned} \mathbf{v} &= \varpi\hat{\boldsymbol{\phi}} + \mathbf{u}, \\ P &= P_0 + \tfrac{1}{2}(\varpi^2 - c^2) + \epsilon\boldsymbol{\gamma}\,.\,\mathbf{r} + p, \end{aligned}\right\} \tag{2.4}$$

where P_0 is the internal pressure at the interface, a constant. Note that the solid rotation assumed ($\hat{\boldsymbol{\phi}}$ is an azimuthal unit vector) is *not* solid corotation, but rotation about the core axis.

The same quantities will be written in container co-ordinates as

$$\left.\begin{aligned} \mathbf{v} &= \bar{\varpi}\hat{\bar{\boldsymbol{\phi}}} + \bar{\mathbf{u}}, \\ P &= P_0 + \tfrac{1}{2}(\bar{\varpi}^2 - c^2) + \epsilon\boldsymbol{\gamma}\,.\,\bar{\mathbf{r}} + \bar{p}. \end{aligned}\right\} \tag{2.5}$$

The governing equations are of the same form in both systems, namely

$$\mathbf{u}_t + \mathbf{u}_\phi + 2\mathbf{k}\times\mathbf{u} + \mathbf{u}\,.\,\nabla\mathbf{u} + \nabla p = E\nabla^2\mathbf{u}, \tag{2.6a}$$

$$\nabla\,.\,\mathbf{u} = 0. \tag{2.6b}$$

(In both equations $\mathbf{u}_\phi$ means partial derivatives of the vector components only; it is the time derivative in a rotating co-ordinate system.) However, the boundary conditions are not identical.

The boundaries relevant to the core co-ordinate expressions are $z = \pm L/2a$ $(= \pm\lambda)$ and $\varpi = c+\eta$. The quantity η represents motion of the free surface away from its equilibrium position. The no-slip condition on the end plates is

$$\mathbf{u}+\varpi\hat{\boldsymbol{\phi}} = \varpi\bar{\hat{\boldsymbol{\phi}}}, \tag{2.7}$$

which can be broken into a normal part

$$w = \mathbf{k}.\mathbf{u} = 0 \tag{2.8}$$

and a tangential part

$$-\mathbf{k}\times\mathbf{k}\times\mathbf{u}+\varpi\hat{\boldsymbol{\phi}} = \varpi\bar{\hat{\boldsymbol{\phi}}}. \tag{2.9}$$

On the free surface there is a kinematic condition

$$\hat{\varpi}.\mathbf{u} = u = D\eta/Dt = \eta_t+\eta_\phi+\mathbf{u}.\nabla\eta, \tag{2.10}$$

a pressure condition

$$p+c\eta+\tfrac{1}{2}\eta^2+\epsilon\boldsymbol{\gamma}.\mathbf{r} = 0, \tag{2.11}$$

and a no-shear-stress condition

$$w_\varpi+u_z = 0 = \frac{1}{\varpi}u_\phi+v_\varpi-\frac{v}{\varpi}. \tag{2.12}$$

Conditions (2.8), (2.10) and (2.11) must be satisfied by any inviscid solution; a viscous solution must satisfy (2.9) and (2.12) as well.

The boundaries relevant to the container co-ordinates are $\bar{z}\,(=z) = \pm\lambda$ and $\bar{\varpi} = 1$, on which the same boundary condition is to be satisfied:

$$\bar{\mathbf{u}} = 0. \tag{2.13}$$

This can be split into normal and tangential parts in an obvious fashion.

Matching of the two representations of the solution will be done away from any of the boundaries. The solution process is such that the relevant equations away from the various boundaries remain the inviscid ($E\to 0$) equations, so that one is to match the Cartesian velocity components.

Matching is not a unique process. A simple method is based on the assertion that u and $\bar{u}$, say, both represent the same function, much as one would do in matching inner and outer expansions in a formal asymptotic matching procedure. Here all that is necessary is to expand the true solution in a Taylor series around the point (x, y) and to note that $\frac{1}{2}\delta$ above and $\frac{1}{2}\delta$ below the point one will obtain $\bar{u}$ and u representations at the same numerical values of ϖ, $\bar{\varpi}$ and ϕ, $\bar{\phi}$.

Let $\psi(x, y)$ denote the true quantity to be matched at some physical point (x, y). At points $\frac{1}{2}\delta$ above and below, the physical quantity can be approximated as

$$\psi(x,y\pm\tfrac{1}{2}\delta) = \psi(x,y)\pm\psi_y(x,y)\times\tfrac{1}{2}\delta+\psi_{yy}(x,y)\times\tfrac{1}{8}\delta^2+\ldots, \tag{2.14}$$

and the difference between these two expressions gives

$$\psi(x,y+\tfrac{1}{2}\delta)-\psi(x,y-\tfrac{1}{2}\delta) = \psi_y(x,y)\,\delta+O(\delta^3). \tag{2.15}$$

Making use of an additional Taylor approximation for the first partial derivative then gives

$$\psi(x,y+\tfrac{1}{2}\delta)-\psi(x,y-\tfrac{1}{2}\delta) = \tfrac{1}{2}[\psi_y(x,y+\tfrac{1}{2}\delta)+\psi_y(x,y-\tfrac{1}{2}\delta)]\,\delta+O(\delta^3). \tag{2.16}$$

If the point (x,y) is midway between two points defined by $\varpi = \overline{\varpi}$, $\phi = \overline{\phi}$, then $\psi(x, y+\frac{1}{2}\delta) = \psi(r,\theta)$ and $\psi(x,y-\frac{1}{2}\delta) = \psi(r,\theta)$, so that (2.16) expresses a matching condition on expressions in core and container co-ordinates evaluated at the same numerical values of their respective arguments. In terms of the cylindrical components of velocity, using (r,θ) to stand for either (ϖ, ϕ) or $(\overline{\varpi}, \overline{\phi})$ as appropriate, the matching conditions are

$$\left.\begin{aligned} u-\overline{u} &= -\tfrac{1}{2}\delta\left[\left(\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)(u+\overline{u})-\frac{1}{r}\cos\theta(v+\overline{v})\right]+O(\delta^3),\\ v-\overline{v} &= -\tfrac{1}{2}\delta\left[\left(\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)(v+\overline{v})+\frac{1}{r}\cos\theta(u+\overline{u})\right]+O(\delta^3).\end{aligned}\right\} \quad (2.17)$$

The question of linearization is not straightforward. In the laboratory both ϵ and E are small, and one must specify what range of parameter space is wanted before one can chose an expansion procedure. Even this proves inadequate for this problem because of the additional degree of freedom provided by the free surface. (This phenomenon is a minor theme of this paper and the reader is asked to be patient. A full exposition appears in § 4.)

One way to decompose the problem is in terms of azimuthal dependence. The equations are of constant-coefficient type in ϕ $(\overline{\phi})$ and one could take a Fourier transform, or perhaps, more simply, keep track of the rectified, singly periodic, etc., components of the system. These will be coupled because of the nonlinearities in the system, but approximate solutions can be obtained by making appropriate assumptions (which limit the range of validity of the solution) regarding ϵ and E.

The dependent variables will be written as

$$\psi = \sum_{n=0}^{\infty} \psi^{(n)}, \quad (2.18)$$

where the index represents the azimuthal wavenumber. *It does not necessarily represent an order, or imply a magnitude.*

The only explicit inhomogeneity in the problem is $\epsilon\boldsymbol{\gamma}.\mathbf{r}$, which is singly periodic in the azimuthal co-ordinate. Thus the singly periodic problem is the logical place to begin, and the assertion that $\mathbf{u}^{(1)} = O(\epsilon)$ is not unreasonable. A simple inviscid solution plus its boundary-layer corrections is not adequate. Because I shall calculate forced rectified flows I need the leading non-zero components of the nonlinear terms, which will be

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbf{u}^{(1)}.\nabla\mathbf{u}^{(1)}\,d\phi.$$

The phase relations are such that the azimuthal component of this term in the interior is zero if one retains only the linear inviscid (Phillips) solution. Additional terms arise from interactions between the Phillips solution and the corrections to the Phillips solution required by boundary-layer sections. These terms will be $O(\epsilon^2E^{\frac{1}{2}})$ and will dominate the next largest terms, which are $O(\epsilon^4)$, if $\epsilon^2 \ll E^{\frac{1}{2}}$. They too prove to be zero.

The entire problem can be solved at once, rather in the manner of applying an Ekman matching condition. One can find solutions independent of z and singly periodic in ϕ. The outer solution satisfies no-slip conditions on $\overline{\varpi} = 1$ and matches the inner solution in the interior. The interior solution satisfies the no-stress condition, and a kinematic condition, on the free surface. The latter conditions are modified by the demand that $\eta^{(1)}$ be identically zero. The result is ten boundary conditions determining the eight arbitrary constants in the solutions to the differential equations, as well as δ and α, in terms of ϵ and E.

The singly periodic problem is taken up in § 3 below. Section 4 includes the axisymmetric part. Section 5 summarizes the results.

3. The singly periodic solution

The largest contribution to the singly periodic nonlinear term arises from interactions between the largest singly periodic term and the largest axisymmetric (or doubly periodic) term. Such a term will be $O(\epsilon^3)$. As the solution I shall find is to be valid only to $O(\epsilon E^{\frac{1}{2}})$, by hypothesis larger, it is justifiable to consider the singly periodic problem as a linear one. Within this section the parenthetical superscript is suppressed in the interest of neatness.

The appropriate linearized equations can be written in component form as

$$\left.\begin{aligned} u_\phi - 2v + p_\varpi &= EDu, \\ v_\phi + 2u + \varpi^{-1}p_\phi &= EDv, \\ p_z &= E\Delta w, \\ \varpi^{-1}(\varpi u)_\varpi + \varpi^{-1}v_\phi + w_z &= 0, \end{aligned}\right\} \tag{3.1}$$

where $D = \Delta - 1/\varpi^2$ and Δ is the scalar Laplacian. The form of the equations is the same in both co-ordinate systems.

Inviscid ($E \to 0$) solutions to these equations may be written in terms of four constants in each co-ordinate system:

$$p = (A_s\varpi + B_s/\varpi)\sin\phi + (A_c\varpi + B_c/\varpi)\cos\phi, \tag{3.2a}$$

$$u = -\tfrac{1}{3}(3A_s + B_s/\varpi^2)\cos\phi + \tfrac{1}{3}(3A_c + B_c/\varpi^2)\sin\phi, \tag{3.2b}$$

$$v = \tfrac{1}{3}(3A_s - B_s/\varpi^2)\sin\phi + \tfrac{1}{3}(3A_c - B_c/\varpi^2)\cos\phi, \tag{3.2c}$$

in core co-ordinates, the expressions in container co-ordinates being identical, with the addition of overbars.

The expressions (3.2) satisfy the normal boundary conditions on the end walls $z = \pm\lambda$ by virtue of the identical vanishing of any axial component. The boundary layers required to satisfy no-slip conditions are non-divergent, as they cancel a non-divergent two-dimensional flow. Those boundary layers, therefore, do not contribute to the determination of the constants, and I shall write them down after I have determined the constants.

The radial boundary-layer equations are very simple (cf. Gans 1970):

$$\tilde{v}_\phi = E\tilde{v}_{\varpi\varpi}. \tag{3.3}$$

No $\tilde{w}$ is required, and $\tilde{u}$ is found from

$$\tilde{u}_\varpi + \varpi^{-1}\tilde{v}_\phi = 0. \tag{3.4}$$

The general solution to (3.3) can be written on $\varpi = c$ as

$$\tilde{v} = \exp\frac{c-\varpi}{(2E)^{\frac{1}{2}}}\left[\tilde{A}\cos\left(\frac{c-\varpi}{(2E)^{\frac{1}{2}}} + \phi\right) + \tilde{B}\sin\left(\frac{c-\varpi}{(2E)^{\frac{1}{2}}} + \phi\right)\right] \tag{3.5}$$

and on $\bar{\varpi} = 1$ as

$$\tilde{\bar{v}} = \exp\frac{\bar{\varpi}-1}{(2E)^{\frac{1}{2}}}\left[\tilde{\bar{A}}\cos\left(\frac{\bar{\varpi}-1}{(2E)^{\frac{1}{2}}}+\bar{\phi}\right)+\tilde{\bar{B}}\sin\left(\frac{\bar{\varpi}-1}{(2E)^{\frac{1}{2}}}+\bar{\phi}\right)\right]. \tag{3.6}$$

The boundary conditions are vanishing shear on $\varpi = c$, i.e.

$$\tilde{v}_\varpi + v_\varpi + c^{-1}u_\phi - c^{-1}v = 0, \tag{3.7}$$

and no-slip on $\bar{\varpi} = 1$, i.e.

$$\tilde{\bar{v}} + \bar{v} = 0. \tag{3.8}$$

After some algebra one obtains the boundary-layer constants in terms of the interior constants:

$$\tilde{\bar{A}} = -\tfrac{1}{3}(3\bar{A}_c - \bar{B}_c), \quad \tilde{\bar{B}} = -\tfrac{1}{3}(3\bar{A}_s - \bar{B}_s), \tag{3.9a, b}$$

$$\tilde{A} = \tfrac{1}{3}(2E)^{\frac{1}{2}}\left[\frac{3}{c}(A_c + A_s) + \frac{1}{c^3}(B_c + B_s)\right], \tag{3.9c}$$

$$\tilde{B} = \tfrac{1}{3}(2E)^{\frac{1}{2}}\left[\frac{3}{c}(A_c - A_s) + \frac{1}{c^3}(B_c - B_s)\right]. \tag{3.9d}$$

Direct integration of (3.4), using (3.5), (3.6) and (3.9), gives boundary values of the radial velocity associated with the boundary layers. These are

$$\tilde{\bar{u}}|_{\bar{\varpi}=1} = -\tfrac{1}{3}(2E)^{\frac{1}{2}}[3(\bar{A}_c - \bar{A}_s) - (\bar{B}_c - \bar{B}_s)]\sin\bar{\phi} + \tfrac{1}{3}(2E)^{\frac{1}{2}}[3(\bar{A}_c + \bar{A}_s) - (\bar{B}_c + \bar{B}_s)]\cos\bar{\phi} \tag{3.10}$$

and

$$\tilde{u}|_{\varpi=c} = \frac{2E}{3c}\left\{\left(\frac{3A_s}{c} + \frac{B_s}{c^3}\right)\sin\phi + \left(\frac{3A_c}{c} + \frac{B_c}{c^3}\right)\cos\phi\right\}. \tag{3.11}$$

A 'closed' problem consists of applying boundary and matching conditions to the general solutions given. Zero tangential velocity and stress on $\bar{\varpi} = 1$ and $\varpi = c$ respectively have been satisfied. There remain the conditions of vanishing normal velocity on $\bar{\varpi} = 1$ and $\varpi = c$ (a consequence of $\eta^{(1)} \equiv 0$), i.e.

$$3\bar{A}_s + \bar{B}_s - (2E)^{\frac{1}{2}}[3(\bar{A}_c + \bar{A}_s) - (\bar{B}_c + \bar{B}_s)] = 0, \tag{3.12a}$$

$$3\bar{A}_c + \bar{B}_c - (2E)^{\frac{1}{2}}[3(\bar{A}_c - \bar{A}_s) - (\bar{B}_c - \bar{B}_s)] = 0, \tag{3.12b}$$

$$\left(3A_s + \frac{1}{c^2}B_s\right) - \frac{2E}{c^2}\left(3A_c + \frac{1}{c^2}B_c\right) = 0, \tag{3.13a}$$

$$\left(3A_c + \frac{1}{c^2}B_c\right) + \frac{2E}{c^2}\left(3A_s + \frac{1}{c^2}B_s\right) = 0, \tag{3.13b}$$

a pressure boundary condition (with $\eta^{(1)} \equiv 0$), i.e.

$$A_s + c^{-2}B_s = \epsilon\cos\alpha, \tag{3.14a}$$

$$A_c + c^{-2}B_c = \epsilon\sin\alpha, \tag{3.14b}$$

and the matching conditions one can derive from (2.17).

The only non-zero matching term on the right-hand side of (2.17) will come from the basic rotations. Other terms with the appropriate symmetries are $O(\epsilon^3)$ and must be neglected to be consistent with the Taylor expansion on which (2.17) is based. It happens that $\delta = O(\epsilon)$, so that such terms should also be neglected to the order to which this segment of the problem is being worked. The matching conditions are then

$$u - \bar{u} = \delta\cos\theta, \quad v - \bar{v} = -\delta\sin\theta, \tag{3.15}$$

which can be rewritten as the four conditions

$$A_s-\bar{A}_s=-\delta,\quad B_s-\bar{B}_s=0,\quad A_c-\bar{A}_c=0,\quad B_c-\bar{B}_c=0. \tag{3.16}$$

The solution procedure is straightforward. Equations (3.16) eliminate the barred constants. Equations (3.13) are a pair of homogeneous algebraic equations for the quantities in parentheses. The determinant does not vanish (it equals $1+4E^2/c^4>0$) so the parenthetical quantities must vanish, which allows the elimination of the B's. Substitution into (3.14) gives the A's in terms of δ and α. Substitution into (3.12) gives the pair of equations

$$[1-(2E)^{\frac{1}{2}}]\,\delta=-\tfrac{1}{2}\epsilon\{(2E)^{\frac{1}{2}}(1+c^2)(\sin\alpha+\cos\alpha)-(1-c^2)\cos\alpha\}, \tag{3.17}$$

$$(2E)^{\frac{1}{2}}\delta=-\tfrac{1}{2}\epsilon(2E)^{\frac{1}{2}}\{(1+c^2)(\sin\alpha-\cos\alpha)-(1-c^2)\sin\alpha\}. \tag{3.18}$$

Neglecting $\sin\alpha$ compared with $\cos\alpha$ and $E^{\frac{1}{2}}$ compared with unity makes it possible to solve (3.17) for δ and (3.18) for $\tan\alpha$. To be consistent one must put $\tan\alpha=\sin\alpha=\alpha$ and $\cos\alpha=1$. One then has

$$\delta=\tfrac{1}{2}\epsilon(1-c^2),\quad \alpha=-2(2E)^{\frac{1}{2}}c^2/(1-c^2), \tag{3.19}$$

and, to the order of approximation appropriate,

$$\left.\begin{aligned}
&A_s=-\tfrac{1}{2}\epsilon,\quad \bar{A}_s=-\tfrac{1}{2}\epsilon c^2,\\
&B_s=\tfrac{3}{2}c^2\epsilon,\quad \bar{B}_s=\tfrac{3}{2}c^2\epsilon,\\
&A_c=c^2\epsilon(2E)^{\frac{1}{2}}/(1-c^2),\quad \bar{A}_c=c^2\epsilon(2E)^{\frac{1}{2}}/(1-c^2),\\
&B_c=-3c^4\epsilon(2E)^{\frac{1}{2}}/(1-c^2),\quad \bar{B}_c=-3c^4\epsilon(2E)^{\frac{1}{2}}/(1-c^2).
\end{aligned}\right\} \tag{3.20}$$

The velocity components in the interior are

$$\left.\begin{aligned}
u&=\tfrac{1}{2}\epsilon\left(1-\frac{c^2}{\varpi^2}\right)\cos\phi+(2E)^{\frac{1}{2}}\epsilon\frac{c^2}{1-c^2}\left(1-\frac{c^2}{\varpi^2}\right)\sin\phi,\\
v&=-\tfrac{1}{2}\epsilon\left(1+\frac{c^2}{\varpi^2}\right)\sin\phi+(2E)^{\frac{1}{2}}\epsilon\frac{c^2}{1-c^2}\left(1+\frac{c^2}{\varpi^2}\right)\cos\phi,
\end{aligned}\right\} \tag{3.21}$$

$$\left.\begin{aligned}
\bar{u}&=\tfrac{1}{4}\epsilon c^2\left(1-\frac{1}{\bar{\varpi}^2}\right)\cos\bar{\phi}+(2E)^{\frac{1}{2}}\epsilon\frac{c^2}{1-c^2}\left(1-\frac{c^2}{\bar{\varpi}^2}\right)\sin\bar{\phi},\\
\bar{v}&=-\tfrac{1}{2}\epsilon c^2\left(1+\frac{1}{\bar{\varpi}^2}\right)\sin\bar{\phi}+(2E)^{\frac{1}{2}}\epsilon\frac{c^2}{1-c^2}\left(1+\frac{c^2}{\bar{\varpi}^2}\right)\cos\bar{\phi}.
\end{aligned}\right\} \tag{3.22}$$

The first terms on the right-hand sides agree with Phillips' solution; the second terms give a viscous correction.

It is a straightforward matter to apply the no-slip boundary conditions on the end walls at this point. The governing differential equations can be taken directly from Gans (1970) and the solutions which cancel the contributions from (3.21) and (3.22) are

$$\begin{aligned}
\tilde{u}=&-\tfrac{1}{2}\epsilon c^2\exp\frac{z-\lambda}{(2E)^{\frac{1}{2}}}\cos\left[\frac{\lambda-z}{(2E)^{\frac{1}{2}}}+\phi\right]+\tfrac{1}{2}\epsilon\frac{c^2}{\varpi^2}\exp\frac{3^{\frac{1}{4}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\left[\cos\frac{3^{\frac{1}{4}}(z-\lambda)}{(2E)^{\frac{1}{2}}}+\phi\right]\\
&-(2E)^{\frac{1}{2}}\frac{c^2}{1-c^2}\exp\frac{z-\lambda}{(2E)^{\frac{1}{2}}}\sin\left[\frac{\lambda-z}{(2E)^{\frac{1}{2}}}+\phi\right]\\
&+(2E)^{\frac{1}{2}}\frac{\epsilon c^2}{1-c^2}\frac{c^2}{\varpi^2}\exp\frac{3^{\frac{1}{4}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\sin\left[\frac{3^{\frac{1}{4}}(z-\lambda)}{(2E)^{\frac{1}{2}}}+\phi\right],
\end{aligned} \tag{3.23}$$

$$\tilde{v} = \tfrac{1}{2}\epsilon c^2 \exp\frac{z-\lambda}{(2E)^{\frac{1}{2}}}\sin\left[\frac{\lambda-z}{(2E)^{\frac{1}{2}}}+\phi\right]+\tfrac{1}{2}\epsilon\frac{c^2}{\varpi^2}\exp\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\sin\left[\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}+\phi\right]$$
$$-(2E)^{\frac{1}{2}}\frac{\epsilon c^2}{1-c^2}\exp\frac{z-\lambda}{(2E)^{\frac{1}{2}}}\cos\left[\frac{\lambda-z}{(2E)^{\frac{1}{2}}}+\phi\right]$$
$$-(2E)^{\frac{1}{2}}\frac{\epsilon c^2}{1-c^2}\frac{c^2}{\varpi^2}\exp\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\cos\left[\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}+\phi\right]. \tag{3.24}$$

The expressions in container co-ordinates are identical; one needs merely to replace (ϖ, ϕ) by $(\bar{\varpi}, \bar{\phi})$. It is straightforward matter to show that (3.23) and (3.24) represent the solution on $z = -\lambda$ upon replacement of $\lambda - z$ by $z+\lambda$. As stated above $\nabla . \tilde{\mathbf{u}} = 0$.

4. The axisymmetric problem

The interior and the Ekman layers

The governing differential equations are just the azimuthally averaged full equations with the nonlinear term truncated as in § 2 above, namely

$$\overline{\mathbf{u}.\nabla\mathbf{u}} \approx \overline{\mathbf{u}^{(1)}.\nabla\mathbf{u}^{(1)}}.$$

These may be written as

$$-2v^{(0)}+p^{(0)}_{\varpi} = EDu^{(0)}-\hat{\boldsymbol{\varpi}}.\overline{\mathbf{u}^{(1)}.\nabla\mathbf{u}^{(1)}}, \tag{4.1a}$$

$$2u^{(0)} = EDv^{(0)}-\hat{\boldsymbol{\phi}}.\overline{\mathbf{u}^{(1)}.\nabla\mathbf{u}^{(1)}}, \tag{4.1b}$$

$$p^{(0)}_{z} = E\Delta w^{(0)}-\hat{\mathbf{z}}.\overline{\mathbf{u}^{(1)}.\nabla\mathbf{u}^{(1)}}, \tag{4.1c}$$

$$[\varpi u^{(0)}]_{\varpi}+\varpi w^{(0)}_{z} = 0. \tag{4.1d}$$

In the interior the right-hand sides of (4.1*a–c*) are to be calculated using the velocities given by (3.21) or (3.22). The resulting nonlinear term will have components which are $O(\epsilon^2)$, $O(E^{\frac{1}{2}}\epsilon^2)$ and $O(E\epsilon^2)$. What is desired is the leading non-zero term in each equation. For the radial equation this term is the one that would be calculated from Phillips' solution, $-\epsilon^2c^4/2\varpi^5$. All three contributions vanish in the azimuthal direction. Thus the interior radial velocity must be small compared with $E\epsilon^2$, which I shall replace by zero in writing the interior version of (4.1).

Keeping only the leading terms in the interior version of (4.1) gives

$$\left.\begin{aligned} -2v^{(0)}+p^{(0)}_{\varpi} &= -\tfrac{1}{2}\epsilon^2c^4/\varpi^5,\\ 2u^{(0)} &= 0\end{aligned}\right\} \tag{4.2}$$

in both co-ordinate systems. The solution is

$$p^{(0)} = 2\int v^{(0)}+\tfrac{1}{8}\epsilon^2c^4/\varpi^4, \tag{4.3}$$

where $v^{(0)}$ can be any function of the radial co-ordinate.

The inviscid problem cannnot determine $v^{(0)}$. It is determined by a global mass balance criterion, which is equivalent to requiring the Ekman layers on $z = \pm\lambda$ to be non-divergent. To proceed one must solve an inhomogeneous Ekman-layer problem.

The right-hand side of the boundary-layer momentum equation has non-zero azimuthal as well as radial components. However, because of the cross-coupling made possible by the non-vanishing viscous terms, it is not necessary to retain the leading non-zero term in both equations. The radial interaction is $O(\epsilon^2)$ and the azimuthal $O(E^{\frac{1}{2}}\epsilon^2)$, so the latter can be neglected in comparison with the former, making the relevant equations

$$\left.\begin{aligned} -2v^{(0)}-E\tilde{u}^{(0)}_{zz} &= \frac{\epsilon^2c^4}{2\varpi^5}\left[\exp\frac{2\times 3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}-2\cos\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\exp\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\right],\\ 2\tilde{u}^{(0)}-E\tilde{v}^{(0)}_{zz} &= 0,\end{aligned}\right\} \tag{4.4}$$

and the general solution can be written as

$$\tilde{u}^{(0)} = v^{(0)}\exp\frac{z-\lambda}{E^{\frac{1}{2}}}\sin\frac{z-\lambda}{E^{\frac{1}{2}}}+\frac{\epsilon^2c^4}{20\varpi^5}\left\{\exp\frac{z-\lambda}{E^{\frac{1}{2}}}\left[3\cos\frac{z-\lambda}{E^{\frac{1}{2}}}-17\sin\frac{z-\lambda}{E^{\frac{1}{2}}}\right]\right.$$
$$\left.-3\exp\frac{2\times 3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}+24\exp\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\right\}, \tag{4.5}$$

$$\tilde{v}^{(0)} = -v^{(0)}\exp\frac{z-\lambda}{E^{\frac{1}{2}}}\cos\frac{z-\lambda}{E^{\frac{1}{2}}}+\frac{\epsilon^2c^4}{20\varpi^5}\left\{\exp\frac{z-\lambda}{E^{\frac{1}{2}}}\left[17\cos\frac{z-\lambda}{E^{\frac{1}{2}}}+3\sin\frac{z-\lambda}{E^{\frac{1}{2}}}\right]\right.$$
$$\left.-\exp\frac{2\times 3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}-16\exp\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\cos\frac{3^{\frac{1}{2}}(z-\lambda)}{(2E)^{\frac{1}{2}}}\right\}. \tag{4.6}$$

The same result, with $\lambda-z$ replaced by $\lambda+z$, is obtained on $z=-\lambda$.

In the steady state the mass within any fixed volume must be constant, or, equivalently, the total flux across any fixed surface must be zero. The mass flux across the surface $\varpi = r$ is given by

$$\int_0^{2\pi}\int_{-\lambda}^{\lambda} u(r)\,r\,d\phi\,dz, \tag{4.7}$$

where $u(r)$ is the radial component of the velocity. The integration with respect to ϕ is equivalent to the averaging process, so that

$$\int_0^{2\pi}\int_{-\lambda}^{\lambda} u(r)\,r\,d\phi\,dz = 2\pi r\int_{-\lambda}^{\lambda} u^{(0)}(r)\,dz = 0, \tag{4.8}$$

and the leading term of $u^{(0)}(r)$ is the boundary-layer part; the discussion above has established that the interior mass transport is less than $E\epsilon^2$, while the boundary-layer mass transport can be $O(E^{\frac{1}{2}}\epsilon^2)$.

Insertion of (4.5) (and its mirror image on $z=-\lambda$) into (4.8) and subsequent integration determines the interior azimuthal velocity profile:

$$v^{(0)} = -\frac{\epsilon^2c^4}{\varpi^5}\left(\frac{27(\frac{2}{3})^{\frac{1}{2}}-20}{20}\right). \tag{4.9}$$

Here the numerical factor in brackets is 0·102270384...; where convenient it will be denoted by N.

It should be noted that the vanishing of the mass flux in the boundary layers is a more stringent requirement than asking that the divergence from the Ekman layers be zero. The latter requirement permits an additional potential vortex flow in the interior, however the mass flux criteria show that this vortex must be small compared with $E^{\frac{1}{2}}\epsilon^2$, so that the mass transported will be small compared with $E\epsilon^2$.

The side walls

Both the outer ($\bar{\varpi} = 1$) and the inner ($\varpi = c$) interface is a side wall for the axisymmetric problem, and the usual set of Stewartson layers (Stewartson 1958; see also Greenspan 1968, §2.19) with dimensionless thicknesses $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ are to be expected. Some novelties arise because of the free surface and primary attention will be directed towards these.

The governing equations for the side-wall layers are those given in Greenspan's book, with the addition of an inhomogeneous term in the radial momentum equation. This term is a function of ϖ (or $\bar{\varpi}$) only and can be balanced by a pressure term. Thus, for calculation of velocities the usual homogeneous boundary-layer equations are adequate. In component form

$$\left.\begin{aligned} -2\tilde{v}^{(0)}+\tilde{p}^{(0)}_{\varpi} = 0, \quad 2\tilde{u}^{(0)}-E\tilde{v}^{(0)}_{\varpi\varpi} = 0,\\ \tilde{p}^{(0)}_{z}-E\tilde{w}^{(0)}_{\varpi\varpi} = 0, \quad \tilde{u}_{\varpi}+\tilde{w}_{z} = 0. \end{aligned}\right\} \tag{4.10}$$

The solutions to (4.10) are the Stewartson layers. They are required to satisfy no-slip conditions on $\bar{\varpi} = 1$ and no-shear conditions on $\varpi = c$, as well as having axial transport small compared with $E\epsilon^2$, a condition imposed by the inability of the Ekman layer to accept the transport.

The form of the solutions is the same on both boundaries. On the inner boundary

$$\tilde{u}^{(0)} = \tfrac{1}{2}E\gamma_0^2 W_0 \exp\gamma_0(c-\varpi)+\tfrac{1}{2}E\sum_{n=1}^{\infty}\left[\sum_{i=1}^{3}\gamma_{ni}^2 W_{ni}\exp\gamma_{ni}(c-\varpi)\right]\cos\frac{n\pi z}{\lambda}, \tag{4.11a}$$

$$\tilde{v}^{(0)} = W_0\exp\gamma_0(c-\varpi)+\sum_{n=1}^{\infty}\left[\sum_{i-1}^{3}W_{ni}\exp\gamma_{ni}(c-\varpi)\right]\cos\frac{n\pi z}{\lambda}, \tag{4.11b}$$

$$\begin{aligned}\tilde{w}^{(0)} &= \tfrac{1}{2}\gamma_0^3 EW_0 z\exp\gamma_0(c-\varpi)\\ &\quad+\sum_{n=1}[W_{n1}\exp\gamma_{n1}(c-\varpi)-W_{n2}\exp\gamma_{n2}(c-\varpi)-Wn_3\exp\gamma_{n3}(c-\varpi)]\sin\frac{n\pi z}{\lambda},\end{aligned} \tag{4.11c}$$

where

$$\gamma_0 = (\lambda^2E)^{-\frac{1}{4}}, \quad \gamma_{n1} = (\lambda E/2n\pi)^{-\frac{1}{3}}, \quad \gamma_{n2} = e^{\frac{1}{3}i\pi}\gamma_{n1}, \quad \gamma_{n3} = e^{-\frac{1}{3}i\pi}\gamma_{n1}, \tag{4.12}$$

and W_0 and the W_{ni} are constants.

These equations have been derived under the same general assumptions as those in §2.19 of Greenspan's book, and are the axisymmetric equivalent, with the origin of z and the Ekman number suitably redefined, of the equations he gives. By choosing the form (4.11), the differential equations and the Ekman compatibility condition have been satisfied, and the solutions decay exponentially away from the free surface. A very similar set of equations can be written down for the outer wall.

The boundary conditions are that the shear on the interface vanishes and that there be no flux out of these boundary layers at their ends. The latter condition can be made more stringent. As there is no interior radial velocity, one can demand that the axial mass flux be zero at any axial position. This is physically equivalent to asking that the radial velocity component should vanish at the edge of the boundary layer.

The azimuthal shear condition is divided into two by its z dependence, i.e.

$$\gamma_0 W_0 = 6N\epsilon^2/c^2, \tag{4.13a}$$

$$\gamma_{n1}W_{n1}+\gamma_{n2}W_{n2}+\gamma_{n3}W_{n3} = 0; \tag{4.13b}$$

the axial shear condition is

$$\tfrac{1}{2}\gamma_0^4 E W_0 z + \sum_{n=1}^{\infty} (\gamma_{n1} W_{n1} - \gamma_{n2} W_{n2} - \gamma_{n3} W_{n3}) \sin\frac{n\pi z}{\lambda} = 0; \tag{4.14}$$

the mass flux condition is

$$\tfrac{1}{2}\gamma_0^2 E W_0 z + \sum_{n=1}^{\infty} \left(\frac{W_{n1}}{\gamma_{n1}} - \frac{W_{n2}}{\gamma_{n2}} - \frac{W_{n3}}{\gamma_{n3}}\right) \sin\frac{n\pi z}{\lambda} = 0. \tag{4.15}$$

Equations (4.14) and (4.15) are solved by a Fourier decomposition. While one's intuition is that this process may be in conflict with the basis boundary-layer assumption, R. Whiting (private communication) has calculated some of the resulting series and finds then to be slowly varying functions of z.

Solution of (4.13)–(4.15), retaining in all cases only the leading terms, gives

$$\left.\begin{aligned} W_0 &\sim 6N(\lambda^2 E)^{\frac{1}{4}} \epsilon^2/c^2 = O(E^{\frac{1}{4}}\epsilon^2). \\ W_{n1} &\sim \tfrac{1}{2}\lambda^2 \gamma_{n1}^{-1} \Gamma_n W_0 = O(E^{\frac{7}{12}}\epsilon^2), \\ W_{n2} &\sim \frac{\gamma_{n3}\lambda\Gamma_n E^{\frac{1}{2}}}{2i \sin\frac{2}{3}\pi} W_0 = O(E^{\frac{5}{12}}\epsilon^2), \\ W_{n3} &\sim -\frac{\gamma_{n2}\lambda\Gamma_n E^{\frac{1}{2}}}{2i\sin\frac{2}{3}\pi W_0} = O(E^{\frac{5}{12}}\epsilon^2), \end{aligned}\right\} \tag{4.16}$$

where $\Gamma_n = (\lambda \cos n\pi)/n\pi$, and the neglected terms are factors of $E^{\frac{1}{6}}$ smaller.

The azimuthal velocity in the boundary layer is small compared with that in the interior; this is to be expected when one is balancing stresses. The axial velocity is of some interest. It is of magnitude $E^{\frac{1}{2}}\epsilon^2$ in the $E^{\frac{1}{4}}$ layer and $E^{\frac{5}{12}}\epsilon^2$ in the $E^{\frac{1}{3}}$ layer. The flux in each layer is thus $E^{\frac{3}{4}}\epsilon^2$, and must cancel. There is a circulation within the double Stewartson structure: towards the end plates in the $E^{\frac{1}{4}}$ layer and returning in the $E^{\frac{1}{3}}$ layer.

5. Discussion

The basic features of the flow I have calculated are quickly summarized. There is a non-axisymmetric interior circulation which has the net effect of displacing the air column a distance $O(\epsilon)$ down and a distance $O(E^{\frac{1}{2}}\epsilon)$ in the direction $\mathbf{k} \times \mathbf{g}$. Associated with the offset are various shearing motions against the boundaries and interior advections of momentum. The major effect of these is to induce an interior swirl velocity proportional to the inverse fifth power of the radius. There is no axisymmetric pumping in the interior, but there is circulation within and between the two Stewartson layers on the radial boundaries.

I have been unable to find in the literature experimental data with which to make comparisons. The only data on retrograde rotation are those given by Greenspan (1976) for a rigid straw. From his figure 4 it appears that typical retrograde rotation rates are 10–20 % of the basic rotation rate for c in the range 0·025–0·053. Equation (4.9) of this paper gives retrograde rotation rates of $N\alpha^2$, where $\alpha = \epsilon/c$. [By retrograde rotation I mean the difference between the observed rotation of the interface and the actual rotation of the container. In Greenspan's dimensional notation this is $\Omega - \Omega_s$ and the comparison should be between Greenspan's $(\Omega - \Omega_s)/\Omega$ and this paper's $v^{(0)}/c$, calculated from (4.9).] Greenspan's observations appear to lie between his theoretical curves, which ignore end effects, and mine, which assume a different interface condition.

Phillips' physical instability criterion, that the radial pressure gradient reverses sign, is surely a sufficient condition for instability. The radial component of the momentum equation can be 'solved' for the radial pressure gradient, and its minimum value, obtained at $\phi = \frac{1}{2}\pi$, is

$$\min |\partial p/\partial \varpi| = c - 3\epsilon - \tfrac{1}{2}(2N+1)\,\epsilon^2/c. \tag{5.1}$$

In terms of Greenspan's $\alpha\,(=\epsilon/c)$, the critical $p_\varpi = 0$ condition can be written as

$$0 = \tfrac{1}{2}[1 - 3\alpha - \tfrac{1}{2}(2N+1)\,\alpha^2]. \tag{5.2}$$

This criterion reduces to $\alpha = \frac{1}{3}$ in the linear (Phillips) case and $\alpha = 0{\cdot}314$ in the case considered here.

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